Pairing in the quantum Hall system

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We find an analogy between the single skyrmion state in the quantum Hall system and the BCS superconducting state and address that the quantum mechanical origin of the skyrmion is electronic pairing. The skyrmion phase is found to be unstable for magnetic fields above the critical field $B_c(T)$ at temperature T, which is well represented by the relation $B_c(T)/B_c(0) \approx [1-(T/T_c)^3]^{1/2}$.

In the quantum Hall system,[1] skyrmions, i.e., charged spin textures, have been attracting much attention. When Barret and his coworkers found that the electron spin polarization drops precipitously on either side of $\nu = 1$ ($\nu \equiv N/N_\phi$, where N is the total number of electrons and N_ϕ is the Landau level degeneracy),[2] the ground state was thought to consist of skyrmion which was already theoretically studied through the nonlinear sigma model[3] or the Hartree-Fock approach for zero temperature.[4] Later, the skyrmion was described by a microscopic model without using the sigma model,[5] however, electron-electron interactions were assumed to be unrealistic point interactions. From finite-size calculations for the electron-electron Coulomb interactions, Xie and He[6] found the lowest energy states, which are nearly degenerated but have different total spins, in weak Zeeman coupling limit. These low-lying states were used to describe skyrmions, which were suggested to be symmetry-breaking states.[6,7] However, it is still desirable to know the quantum mechanical origin of skyrmions. Here we find that the quamtum Hall system has a pairing effect, similar to superconducting systems. Since the BCS theory has been very successful in superconductivity,[8] we may use the same theoretical approach to understand the role of electron-electron interactions in the quantum Hall system.

In this work, we present a picture that the skyrmion state consists of electron-hole pairs, as the BCS superconducting state made of the Cooper pairs. Without the nonlinear sigma model, we are able to derive the skyrmion wave function near the filling factor $\nu = 1$, which is identical to that of Fertig and his coworkers,[4] from the eigenstates of the mean-field pairing Hamiltonian. We find that the skyrmion phase is unstable for temperatures or magnetic fields above their critical values.

Let us consider a quantum Hall system of N electrons with the lowest Landau level degeneracy $N_\phi = N + 1$, which has the filling factor ($\nu = 1 - \epsilon$) slightly less than one. The system with $\nu = 1+\epsilon$ can be also analyzed using the well known particle-hole symmetry. The quantum Hall sample is assumed to have no impurities, so that the skyrmion size is macroscopic in weak Zeeman coupling limit. For sufficiently high magnetic fields, the ground state of the system will be in a state Ψ_0^+ with all the spins aligned along the same up direction and one hole at the origin. When the applied magnetic field is reduced to lower the Zeeman energy, Ψ_0^+ becomes unstable against the creation of an electron-hole pair, which is described by $c^\dagger_{m,\downarrow} c_{m+1,\uparrow}|\Psi_0^+\rangle$, where $c^\dagger_{m,\sigma}$ and $c_{m,\sigma}$ are the creation and annihilation operators for the electron in the angular momentum $l = m$ and spin σ state. Then, the single pair states with different m's can be linearly combined to form the new ground state or the low-lying excited states of the system.

In analogy with the BCS pairing Hamiltonian, there exists an interaction potential V representing *pair-pair* correlations, which are responsible for the rigidity of the pair condensate. By noting that electron-electron interactions are purely the Coulomb interactions in our theory, the potential energy V in the Hamiltonian can be written as

$$V = \sum_{m \neq n} V_{m,n+1,n,m+1} c^\dagger_{m,\downarrow} c^\dagger_{n+1,\uparrow} c_{m+1,\uparrow} c_{n,\downarrow}, \qquad (1)$$

where V_{m_1,m_2,m_3,m_4} is the Coulomb interaction matrix between the single particle states with $l = m_1, m_2, m_3$, and m_4 in the 2-dimensional system.[9] As in the case of superconductivity, the electron-hole pairings can lead to a phase transition, which is called a *skyrmionic transition* (this notion will be clearly discussed later). In this transition, the topological charge Q of the state changes abruptly from -1 to 0 at the critical temperature T_c, which depends on the strength of the Zeeman coupling. Note that although the Coulomb interaction is purely repulsive, the expectation value of V can be negative, which makes the skyrmion phase stable. Using the Hartree-Fock decomposition, one can easily show that the peculiar form of V allows for the scattering between the states with different total spins (S_z), but, it conserves $L_z - S_z$, where L_z is the z-component of the total orbital angular momentum. The conservation of $L_z - S_z$ ensures that V represents proper interactions for the low-lying states of the system, because as found in Ref. 6, there are almost degenerate lowest-energy states with the same value of $L_z - S_z$, which are well separated from higher energy states. Similarly, using the particle-hole symmetry, our theory can also be extended to the case of $\nu = 1 + \epsilon$, where the wave function has $L_z + S_z$ symmetry and Q changes from 1 to 0 at the transition.[7]

To calculate the physical quantities, we approximate all the interactions by their thermal averages except for V in the Hamiltonian. The problem is then reduced to finding the properties of the reduced Hamiltonian $H_{red} = \sum_{m,\sigma} \epsilon_{m,\sigma} c^\dagger_{m,\sigma} c_{m,\sigma} + V$, where $\epsilon_{m,\sigma}$ is the self-energy for the electron in the state $|m, \sigma\rangle$. In a mean-field concept, we ignore the fluctuations about the expectation values of $c^\dagger_{m+1\uparrow} c_{m\downarrow}$. This suggests that it will be useful to express such a product of operators formally as $(c^\dagger_{m+1,\uparrow} c_{m,\downarrow} - b_m) + b_m$, and subsequently neglect quantities which are bilinear in the presumably small fluctuation term in parentheses. If we follow this procedure with our reduced Hamiltonian, we obtain a model Hamiltonian

$$H_M = \sum_{m,\sigma} \epsilon_{m,\sigma} c^\dagger_{m,\sigma} c_{m,\sigma} - \sum_m (\Delta^*_m c^\dagger_{m+1,\uparrow} c_{m,\downarrow} + \Delta_m c^\dagger_{m,\downarrow} c_{n+1,\uparrow} + \Delta^*_m b_m), \quad (2)$$

where b_m's are to be determined self-consistently from the relation $b_m = \langle c^\dagger_{m+1,\uparrow} c_{m,\downarrow}\rangle$ and Δ_m's are the order parameters defined as

$$\Delta_m = \sum_n V_{m,n+1,n,m+1} \langle c^\dagger_{n+1,\uparrow} c_{n,\downarrow}\rangle. \quad (3)$$

As was done by the Bogoliubov transformation in the BCS theory, the Hamiltonian in Eq. (2) can be diagonalized by a suitable linear transformation which is specified by

$$\begin{aligned} c^\dagger_{m,\downarrow} &= v_m \gamma^\dagger_{m,0} + u^*_m \gamma^\dagger_{m,1} \\ c_{m+1,\uparrow} &= -u^*_m \gamma_{m,0} + v_m \gamma_{m,1}, \end{aligned} \quad (4)$$

where the coefficients u_m and v_m satisfy $|u_m|^2 + |v_m|^2 = 1$ and $\gamma_{m,0}$ and $\gamma_{m,1}$ are the new Fermi operators. Using these new operators, the model Hamiltonian in Eq. (2) is expressed as

$$H_M = -\sum_m \Delta_m b^*_m + \sum_m E_m (\gamma^\dagger_{m,0}\gamma_{m,0} - \gamma^\dagger_{m,1}\gamma_{m,1}) + \sum_m \eta_m (\gamma^\dagger_{m,0}\gamma_{m,0} + \gamma^\dagger_{m,1}\gamma_{m,1}), \quad (5)$$

where $\eta_m = (\epsilon_{m+1,\uparrow} + \epsilon_{m,\downarrow})/2$, $E_m = \sqrt{\xi^2_m + \Delta^2_m}$, and $\xi_m = (\epsilon_{m,\downarrow} - \epsilon_{m+1,\uparrow})/2$. Here E_m and ξ_m satisfy the following relations;

$$|v_m|^2 = 1 - |u_m|^2 = \frac{1}{2}(1 - \frac{\xi_m}{E_m}). \quad (6)$$

If we measure the energies with respect to the chemical potential lying between the up- and down-spin levels, we can ignore the last term in Eq. (5) because η_m is negligibly small, as compared to E_m.

The ground state $|\Psi_G\rangle$ of H_M is then written as

$$|\Psi_G\rangle = \prod_m \gamma^\dagger_{m,1}|0\rangle = \prod_m (u_m c^\dagger_{m,\downarrow} + v_m c^\dagger_{m+1,\uparrow})|0\rangle, \quad (7)$$

where $|0\rangle$ is the vacuum state, and the ground state energy which is evaluated analytically[4,7] is found to be lower than those for the states with $\Delta_m = 0$. Note that the ground state is just equivalent to the skyrmion wavefunction introduced in Ref. 4. This result suggests that *the quantum mechanical origin of the skyrmion in the quantum Hall system is electronic pairing.*

Since the Hamiltonian H_M in Eq. (5) describes independent excitations and the γ_m operators obey Fermi statistics, the probability of the excitation in thermal equilibrium is given by $\langle \gamma^\dagger_{m,0}\gamma_{m,0}\rangle = 1 - \langle \gamma^\dagger_{m,1}\gamma_{m,1}\rangle = f(E_m)$, where $f(E_m)$ is the usual Fermi function. Using this relation and the linear transformation in Eq. (4), Eq. (3) can be rewritten as

$$\begin{aligned} \Delta_m &= {\sum_n}' V_{m,n+1,n,m+1} u^*_n v_n \langle \gamma^\dagger_{m,1}\gamma_{m,1} - \gamma^\dagger_{m,0}\gamma_{m,0}\rangle \\ &= \frac{1}{2} {\sum_n}' V_{m,n+1,n,m+1} \frac{\Delta_n}{E_n} \tanh \frac{1}{2}\beta E_n, \end{aligned} \quad (8)$$

where $\beta = 1/k_B T$ and the prime denotes $m \neq n$. In the first equality in Eq. (8), the expectation values of the off-diagonal terms such as $\gamma^\dagger_{m,0}\gamma_{m,1}$ are neglected because their contributions are not important in thermal equilibrium. By solving self-consistently the order parameter equation, we can obtain Δ_m as a function of T for each magnetic field and find that the order parameters have non-zero values only when $B < B_c$ and $T < T_c$, where B_c and T_c are the critical magnetic field and critical temperature, respectively. Since there is an effective positive charge in the vicinity of the origin, ξ_m has in fact the dependence of m, where m's are small integer numbers. For computational convenience, assuming $\xi_m = \xi$ for all values of m, we can determine ξ in unit of the typical Coulomb energy $e^2/\epsilon l$ from the self-consistent equation $\xi = \tilde{g} + E_{ex} \tanh(\beta\xi/2)$, where $\tilde{g} = \frac{1}{2} g^* \mu_B B/(e^2/\epsilon l)$ is the Zeeman coupling parameter, $l = \sqrt{\hbar c/eB_\perp}$ is the magnetic length, and B and $B_\perp$ are the total applied mgnetic field and its normal componenet to the 2D plane, respectively. Here g^* and ϵ are the band effective g-factor and the dielectric constant of the host material in the quantum Hall sample. To determine the unknown parameter E_{ex} that comes from the exchange interaction, we first calculate the critical value of ξ from Eq. (8) at T = 0, which gives zero values for the order parameters Δ_m. Using the known critical value ($\tilde{g}_c = 0.0265$) for the Zeeman coupling parameter,[7] at which the skyrmion state disappears, the value of E_{ex} is determined so that the critical value ξ_c is equivalent to $\tilde{g}_c + E_{ex}$ at T = 0. Then, this value of E_{ex} is used for the self-consistent solutions of Eq. (8) at finite temperatures. In fact, our approach of using E_{ex} as a parameter and a constant value for ξ_m change slightly the magnitudes of Δ_m, however, it does not affect the main result of our calculations.

Once $\Delta_m(T)$ is calculated, the temperature-dependent excitation energy E_m and the occupation number $f_m =$

$f(E_m)$ can be determined. Then, the specific heat is calculated by $C = TdS/dT$, where S is the electronic entropy of a fermion gas, defined as $S = -2k_B \sum_m [(1 - f_m)\ln(1 - f_m) + f_m \ln f_m]$. In the 2D electron system with the magnetic field perpendicular to the plane, since there is no Meissner effect, the system has no latent heat and the entropy is continuous at T_c. Fig. 1 shows the discontinuity of the specific heat at T_c, and the size of the discontinuity is readily evaluated as follows;

$$\Delta C = (C_s - C_n)\Big|_{T_c} = -\frac{f(\xi_c)\,[1 - f(\xi_c)]}{k_B T_c}\left(\frac{-d\Delta^2}{dT}\right)\Big|_{T_c}, \quad (9)$$

where $\xi_c = \xi(T_c)$, $\Delta^2 = \sum_m |\Delta_m|^2$, and the subscripts s and n denote the *skyrmion* and *normal* phases, respectively. Here we point out that for a given nonzero $\tilde{g}$, the size of the discontinuity in the specific heat ΔC is finite for a single skyrmion state, however, $\Delta C/C_n$ is negligible in the thermodynamic limit $N \to \infty$. Thus, practically, the observation of the phase transition will be only possible when the Zeeman coupling is very weak so that the skyrmion size becomes macroscopic or many skyrmions contribute to the discontinuity of the specific heat. In the multi-skyrmion case, when we do not take into account inter-skyrmion interactions, the discontinuity of the specific heat at the filling factor ν may be approximated by $\Delta C(\nu) \approx \Delta C N_\phi |\nu - 1|$. It should be addressed that our skyrmion phase transition will not be the second-order phase transition, because no long-range order will exist for a finite-size skyrmion. The skyrmion phase transition involves the localized spins of electrons and is not related to the long-range order, similarly to the Kondo problem, where there appears the quasi-bound state between the conduction-electron spin and the localized impurity spin below T_c.[10]

As already known for zero-temperature systems, there is a critical magnetic field for the *skyrmion-normal* state transition.[7] Based on our pairing theory in the quantum Hall system, we can determine the temperature-dependent critical magnetic field $B_c(T)$, similarly to the the Gorter-Casimir phenomenological theory for superconductors, which gives the relation $B_c(T)/B_c(0) \approx 1 - (T/T_c)^2$. With $B_\perp$ fixed, $B_c(T)$ in our system can be expressed as

$$\frac{\tilde{g}_c(T)}{\tilde{g}_c(0)} = \frac{B_c(T)}{B_c(0)} \approx \left[1 - \left(\frac{T}{T_c}\right)^3\right]^{1/2}, \quad (10)$$

where $k_B T_c$ is calculated to be about $0.24e^2/\epsilon l$ for $\tilde{g} = 0$. This formula for $B_c(T)$ is found to be in good agreement with the results obtained directly from the order parameter equation in Eq. (8) (see Fig. 2). For GaAs samples, T_c for $\tilde{g} \approx 0$ is approximately $12.0\sqrt{B_\perp[\text{Tesla}]}$ K.[11] If fluctuations or impurity effects are considered, the critical temperature is expected to be much lowered, however, this is beyond the scope of our work. Since fluctuations above the mean field is important in two dimensional systems, we expect that the sharp specific heat jump as shown in Fig. 1 will not take place in real systems. The estimation of to what extent $B_c(T)$ and $\Delta C(T)$ are smeared out by the fluctuations, thus, will make our work more explicit as the Ginzburg criterion does in the usual Landau theory. Unfortunately, we are unable to find any reliable criterion for the validity of our mean field theory at this stage, which will be our future work.

In summary, we have presented an analogy between the skyrmion state and the superconducting state. The counterpart of the Cooper pair in superconductivity is found to be the pair which consists of a minority-spin electron and a majority-spin hole. We find the temperature-dependent critical magnetic field $B_c(T)$, above which the skyrmion is unstable at temperature T. Since our approach is based on the mean field approximation and the fluctuations above the mean field are usually important in two dimensional systems, our calculated $B_c(T)$ is considered to be an upper bound for the true temperature-dependent critical magnetic field.

We would like to thank Dr. Y.-J. Kim for useful discussions. This work is supported by the MOST, the CTPC and CMS at Korea Advanced Institute of Science and Technology.

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FIG. 1. The relative specific heat $C_s - C_n$ is drawn as a function of T for the Zeeman coupling parameter $\tilde{g} = 0.01$.

FIG. 2. The temperature-dependent critical magnetic field scaled by $B_c(0)$ is shown as a function of T/T_c, with $B_\perp$ fixed. The dotted and solid lines represent the results from the order parameter equation and the fitting formula 1-$(T/T_c)^3$, respectively.